

# Classical XY Model in 1.99 Dimensions<sup>1</sup>

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## Abstract

We consider the classical XY model ( $O(2)$  nonlinear  $\sigma$ -model) on a class of lattices with the (fractal) dimensions  $1 < D < 2$ . The Berezinskii's harmonic approximation suggests that the model undergoes a phase transition in which the low temperature phase is characterized by stretched exponential decay of correlations. We prove an exponentially decaying upper bound for the two-point correlation functions at non-zero temperatures, thus excluding the possibility of such a phase transition.

The classical XY model (or, equivalently,  $O(2)$  nonlinear  $\sigma$ -model) has been a subject of considerable interest in the contexts of statistical physics and relativistic field theory. The model exhibits a standard ferromagnetic phase transition in dimensions  $d > 2$ , while it undergoes an exotic phase transition called the Berezinskii-Kosterlitz-Thouless transition [1, 2, 3] in  $d = 2$ . The existence of a phase transition in two dimensions is in a remarkable contrast with the  $O(n)$ -rotator models ( $\sigma$ -models) with  $n \geq 3$  such as the classical Heisenberg model, which are conventionally believed to be asymptotically free and have no phase transitions in two dimensions [4]. In one dimension, general arguments guarantee that the XY model (like all the other short range spin systems) has no phase transitions.

It is now common to regard the dimension  $d$  as a continuous parameter, having in mind lattices with fractal structures for example. Then a natural question is whether the classical XY model exhibits a (finite temperature) phase transition in the intermediate dimensions  $1 < d < 2$ . Although there have been no publications directly devoted to this problem as far as we know, we believe that the problem is important and is worth settling.

To illustrate that the problem is nontrivial, denote by  $T_c(d)$  the critical temperature (at which the susceptibility diverges) of the classical XY model on the  $d$ -dimension hypercubic lattice. The rigorously established facts are that  $T_c(d)$  is finite for  $d = 2, 3, 4, \dots$ , and is vanishing for  $d = 1$ . If one naively regards  $T_c(d)$  as a function of the continuous parameter  $d$ , then the most natural “interpolation” may be that  $T_c(d)$  continuously decreases below  $d = 2$ , and vanishes continuously at  $d = 1$  (or some dimension  $1 < d < 2$ ). A much stronger support for this naive guess comes from the observation that a straightforward extension of the Berezinskii's harmonic approximation [1] (which correctly predicts the existence of the Kosterlitz-Thouless transition in  $d = 2$ ) indicates that, in  $1 < d < 2$ , there can be an exotic low temperature phase characterized by stretched exponential decay of correlations. One who came across with this argument might conjecture the existence of a phase transition in  $1 < d < 2$ .

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In the present Letter, we address the above problem and provide a conclusive result. We prove that, in the classical XY model defined on a lattice with the (fractal) dimension  $1 \leq D < 2$ , the two-point correlation function decays exponentially at any finite temperature. Our (fractal) dimension  $D$  coincides with the euclidean dimensions for regular lattices. This result excludes the possibility of the new type of phase transition mentioned above [5]. It also leads us to a somehow unexpected conclusion that  $T_c(D)$ , as a function of continuous dimension  $D$ , drops discontinuously [6] to zero at  $D = 2$ .

The proof is carried out in two steps. First we use the McBryan-Spencer method [7] to prove an upper bound for the two-point correlation function in terms of a stretched exponentially decaying function. This result automatically extends to a very general class of models with continuous symmetries, and strengthens the well-known result that these systems have no long range order in  $d \leq 2$  [8, 9, 10]. Next we apply the Simon's inequality and his argument on the decay of correlation [11] to conclude that the correlation decays exponentially in the classical XY model.

**Definitions:** We consider a general connected lattice  $\Lambda = (\Lambda_s, \Lambda_b)$ , where  $\Lambda_s$  is a set of countably infinite sites  $x, y, \dots$ , and  $\Lambda_b$  is a set of bonds, i.e., pairs of sites  $(x, y), (u, v), \dots$  through which spin variables directly interact. We define the classical XY model on  $\Lambda$ . The Hamiltonian (action) is given by

$$\mathcal{H} = - \sum_{(u,v) \in \Lambda_b} \mathbf{s}_u \cdot \mathbf{s}_v = - \sum_{(u,v) \in \Lambda_b} \cos(\theta_u - \theta_v), \quad (1)$$

where  $\mathbf{s}_x = (\cos \theta_x, \sin \theta_x)$  is the spin variable at site  $x \in \Lambda_s$ . The two-point correlation function at inverse temperature  $\beta \geq 0$  is

$$\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta := Z_\beta^{-1} \left( \prod_z \int_{-\pi}^{\pi} d\theta_z \right) e^{-\beta \mathcal{H}} \cos(\theta_x - \theta_y) \quad (2)$$

with the partition function

$$Z_\beta := \left( \prod_z \int_{-\pi}^{\pi} d\theta_z \right) e^{-\beta \mathcal{H}}. \quad (3)$$

The expectation value (2) is to be interpreted as the thermodynamic limit of the corresponding finite volume quantities.

**Berezinskii's argument:** Before describing our results, we shall briefly review the Berezinskii's argument and discuss its extension. The essence of the Berezinskii's harmonic approximation [1, 7, 12] is to make the model into a Gaussian model by replacing  $\cos(\theta_u - \theta_v)$  in the Hamiltonian (1) by  $[1 - (\theta_u - \theta_v)^2/2]$  and replacing the range of integrals in (2) and (3) to  $(-\infty, \infty)$ . A naive expectation is that the approximation gives sensible results for large  $\beta$

where local difference of fields  $(\theta_u - \theta_v)$  is small [13]. After making the above replacements (and suitably eliminating contributions of a zero-mode), the correlation function is evaluated as

$$\begin{aligned}\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta^{\text{Gauss}} &= \frac{\left( \prod_z \int_{-\infty}^{\infty} d\theta_z \right) e^{-\beta \tilde{\mathcal{H}}} \cos(\theta_x - \theta_y)}{\left( \prod_z \int_{-\infty}^{\infty} d\theta_z \right) e^{-\beta \tilde{\mathcal{H}}}} \\ &= \exp[-\langle (\theta_x - \theta_y)^2 \rangle_\beta^{\text{Gauss}} / 2] = \exp[\beta^{-1} C(x, y) / 2],\end{aligned}\quad (4)$$

where  $\tilde{\mathcal{H}} = \sum_{(u,v) \in \Lambda_b} (\theta_u - \theta_v)^2 / 2 = -\sum_{u \in \Lambda_s} \theta_u (\Delta \theta)_u / 2$  is the approximate Gaussian Hamiltonian. Here the lattice Laplacian is defined as  $(\Delta f)_z = \sum_{y \in \Lambda_s: (y,z) \in \Lambda_b} (f_y - f_z)$ . The quantity  $C(x, y) = -\beta \langle (\theta_x - \theta_y)^2 \rangle_\beta^{\text{Gauss}}$  can be characterized as follows. For fixed  $x, y$ , let  $\varphi_z$  be the solution of the lattice Poisson equation

$$-(\Delta \varphi)_z = \delta_{x,z} - \delta_{y,z}. \quad (5)$$

Then  $C(x, y)$  is equal to the “potential difference”  $C(x, y) = \varphi_y - \varphi_x$ . By examining the solution of (5) in a general  $d$ -dimensional lattice, we find that  $C(x, y)$  behaves for large  $|x - y|$  as

$$C(x, y) \simeq \begin{cases} A(d)|x - y|^{2-d} - B(d) & \text{if } 2 < d \\ -A(d) \log |x - y| & \text{if } d = 2 \\ -A(d)|x - y|^{2-d} & \text{if } 1 \leq d < 2 \end{cases}, \quad (6)$$

where  $A(d)$  and  $B(d)$  are strictly positive constants which depend only on the lattice structure. The asymptotic behavior (6) for nonintegral  $d$  was obtained by extrapolating from those for integral  $d$ . On a fractal lattice, one should interpret  $d$  in (6) as a suitable “fractal dimension” [14].

The approximation (4) along with (6) implies that the asymptotic behavior as  $|x - y| \uparrow \infty$  of the correlation functions for large  $\beta$  (low temperatures) is given by  $\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \simeq \sigma(\beta)^2 > 0$  for  $d > 2$ ,  $\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \approx |x - y|^{-\eta(\beta)}$  for  $d = 2$ , and  $\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \approx \exp[-|x - y|/\xi(\beta)]$  for  $d = 1$ . Here (the approximations for) the order parameter  $\sigma(\beta)$ , the critical exponent  $\eta(\beta)$ , and the correlation length  $\xi(\beta)$  are given by  $\sigma(\beta) = \exp[-B(d)/(4\beta)]$ ,  $\eta(\beta) = A(2)/(2\beta)$ , and  $\xi(\beta) = 2\beta/A(1)$ . These estimates for correlations (including the  $\beta$ -dependence of  $\sigma(\beta)$ ,  $\eta(\beta)$ , and  $\xi(\beta)$ ) recover the known (or expected) behavior in a semi-qualitative manner. It is remarkable that the simple approximation yields such strong results.

In dimensions  $1 < d < 2$ , the same approximation yields the asymptotic behavior

$$\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \approx \exp[-\alpha(\beta)|x - y|^{2-d}], \quad (7)$$

with a positive function  $\alpha(\beta)$  of  $0 \leq \beta < \infty$ , which is usually referred to as a stretched exponential decay. Since the correlation decays exponentially for sufficiently small  $\beta$ , this observation suggests that the classical XY model undergoes a phase transition in  $1 < d < 2$ .

It is interesting that, as we will show in Lemma, the McBryan-Spencer argument leads us rigorously to an upper bound for the correlation decaying by a similar stretched exponential law.

**Main results:** We define the “sphere”  $S_n(x)$  of radius  $n$  centered at  $x \in \Lambda_s$  by

$$S_n(x) := \{y \in \Lambda_s | \text{dist}(x, y) = n\}. \quad (8)$$

Here  $\text{dist}(x, y)$  is the graph-theoretic distance between the sites  $x, y$ , which is defined as the minimum number of bonds in  $\Lambda_b$  that one needs to connect  $x$  and  $y$ . We assume that there exists a “(fractal) dimension”  $D$  of the lattice [15] such that the number of sites in  $S_n(x)$  is bounded uniformly from above as

$$\sup_{x \in \Lambda_s} |S_n(x)| \leq Cn^{D-1} \quad (9)$$

with a positive constant  $C$ . It is obvious that the dimension  $D$  coincides with the euclidean dimension if  $\Lambda$  is a regular lattice.

Then our main result is the following exponentially decaying upper bound for the two-point correlation.

*Theorem*—If the dimension  $D$  satisfies  $1 \leq D < 2$ , the two-point correlation function is bounded as

$$|\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta| \leq \exp[-m(\beta)\{\text{dist}(x, y) - R(\beta)\}] \quad (10)$$

for any  $0 \leq \beta < \infty$ , where  $m(\beta) > 0$  and  $R(\beta) > 0$  are functions of  $\beta$  [16].

In the proof of the theorem, we make use of the following lemma. The lemma states an upper bound for the two-point correlation in terms of a stretched exponentially decaying function like the one obtained from the Berezinskii’s approximation.

We note that the following lemma can be trivially extended to cover more general class of systems with a global continuous symmetry. The examples include  $O(n)$  ( $n > 2$ ) rotators (nonlinear  $\sigma$ -model) [7], quantum Heisenberg models [17], and the Hubbard model [18] for itinerant electrons. The main theorem, on the other hand, can be proved only in the classical XY model for the technical reason that the Simon inequality is known only for one- and two-component spin systems. We suspect that the statement of the theorem is valid for the larger class of models.

*Lemma*—For  $1 \leq D < 2$ , we have

$$|\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta| \leq \exp \left[ -f(\beta)(\text{dist}(x, y)^{2-D} - 1) \right] \quad (11)$$

for any  $0 \leq \beta < \infty$ , where the function  $f(\beta)$  is given by

$$f(\beta) = \max_{q \geq 0} \left[ q - \beta C^2 (\cosh q - 1) / (2 - D) \right] > 0, \quad (12)$$

and is decreasing in  $\beta$ .

**Proof of Lemma:** We fix two sites  $x, y$  throughout the proof. Following McBryan and Spencer [7], we make the complex transformations

$$\theta_z \rightarrow \theta_z + i\phi_z \quad (13)$$

in the numerator of (2), where  $\phi_z$  are real numbers to be determined. This means that we deform the path of integration and use the periodicity of the cosine to cancel contributions of the lateral contours. The transformation combined with  $|e^{iz}| = 1$  for real  $z$  yields

$$\begin{aligned} |\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta| &\leq e^{-(\phi_x - \phi_y)} Z_\beta^{-1} \left( \prod_z \int_{-\pi}^{\pi} d\theta_z \right) \exp \left[ \beta \sum_{(u,v) \in \Lambda_b} \cos(\theta_u - \theta_v) \cosh(\phi_u - \phi_v) \right] \\ &\leq e^{-(\phi_x - \phi_y)} \exp \left[ \beta \sum_{(u,v) \in \Lambda_b} (\cosh(\phi_u - \phi_v) - 1) \right]. \end{aligned} \quad (14)$$

Following the idea of Picco [19], we choose  $\{\phi_z\}$  as

$$\phi_z = \begin{cases} q(N^{2-D} - 1) & \text{if } z = x \\ q(N^{2-D} - \text{dist}(x, z)^{2-D}) & \text{if } 1 \leq \text{dist}(x, z) \leq N \\ 0 & \text{if } \text{dist}(x, z) > N \end{cases} \quad (15)$$

where  $N = \text{dist}(x, y)$ , and  $q$  is a positive constant to be determined. From the choice (15), we have that

$$\phi_x - \phi_z = 0 \quad (16)$$

for  $z$  such that  $\text{dist}(x, z) = 1$ , and that

$$|\phi_u - \phi_v| \leq qn^{1-D} \leq q \quad (17)$$

for  $u, v$  satisfying  $n = \text{dist}(x, u) = \text{dist}(x, v) - 1 \neq 0$ .

The summation in the exponential in the right-hand side of (14) can be evaluated as

$$\begin{aligned} &\sum_{(u,v) \in \Lambda_b} (\cosh(\phi_u - \phi_v) - 1) \\ &= \sum_{n=1}^{N-1} \sum_{u: \text{dist}(x, u)=n} \sum_{\substack{v: (u,v) \in \Lambda_b \\ \text{dist}(x, v) \geq \text{dist}(x, u)}} [\cosh(\phi_u - \phi_v) - 1] \\ &\leq \sum_{n=1}^{N-1} \sum_{u: \text{dist}(x, u)=n} \sum_{\substack{v: (u,v) \in \Lambda_b \\ \text{dist}(x, v) \geq \text{dist}(x, u)}} (\cosh q - 1) q^{-2} (\phi_u - \phi_v)^2 \end{aligned}$$

$$\begin{aligned}
&\leq C(\cosh q - 1) \sum_{n=1}^{N-1} \sum_{u: \text{dist}(x,u)=n} n^{2-2D} \\
&\leq C^2(\cosh q - 1) \sum_{n=1}^{N-1} n^{D-1} \times n^{2-2D} \\
&\leq C^2(\cosh q - 1)(N^{2-D} - 1)/(2 - D),
\end{aligned} \tag{18}$$

where we have used (16) and the choice (15) of  $\{\phi_z\}$  to rewrite the summation at the first line. The succeeding three bounds have been obtained by using (17) and (9). By substituting (18) and (15) into the main bound (14), we get a bound of the form (11). Finally we optimize the bound by choosing the constant  $q$  according to (12).

**Proof of Theorem:** We assume  $1 < D < 2$ . Recall the Simon inequality [11] for the classical XY model

$$\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \leq \beta \sum_{\substack{u \in V, v \notin V \\ (u,v) \in \Lambda_b}} \langle \mathbf{s}_x \cdot \mathbf{s}_u \rangle_\beta \langle \mathbf{s}_v \cdot \mathbf{s}_y \rangle_\beta, \tag{19}$$

where the finite set  $V$  is chosen so that  $x \in V$ , and  $y \notin V$ . We set  $V = S_R(x)$  with some  $R < \text{dist}(x, y)$ . By using (9) and the bound (11) of the lemma, (19) implies

$$\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \leq e^{-m(\beta)R} \langle \mathbf{s}_{v_1} \cdot \mathbf{s}_y \rangle_\beta \tag{20}$$

with

$$e^{-m(\beta)R} = C^2 \beta R^{D-1} \exp \left[ -f(\beta)(R^{2-D} - 1) \right], \tag{21}$$

where the site  $v_1$  gives the maximum value of  $\langle \mathbf{s}_v \cdot \mathbf{s}_y \rangle_\beta$  in the summation in the right-hand side of (19). Noting that the right-hand side of (21) is less than 1 for sufficiently large  $R$ , we let  $R(\beta)$  be the value of  $R$  which maximizes the “mass”  $m(\beta)$  in the left-hand side of (21). We note that, for such  $R(\beta)$  and for any given distance  $r = \text{dist}(x, y)$ , there exist non-negative integers  $\ell$  and  $r'$  satisfying  $r = R(\beta)\ell + r'$  and  $r' < R(\beta)$ . (For  $r < R(\beta)$ , we set  $r' = r$  and  $\ell = 0$ .) By setting  $R = R(\beta)$ , we can apply the Simon inequality (19) to the right-hand side of (20) repeatedly at least  $(\ell - 1)$  more times. We thus obtain the desired exponentially decaying bound [20] as

$$\langle \mathbf{s}_x \cdot \mathbf{s}_y \rangle_\beta \leq e^{-m_\beta R(\beta)\ell} \langle \mathbf{s}_{v_\ell} \cdot \mathbf{s}_y \rangle_\beta \leq e^{-m_\beta(r-r')}. \tag{22}$$

**Note added<sup>2</sup>:** After the publication of the paper, we learned that the XY model on a class of finitely ramified fractals is studied in [21], which is one of the pioneering papers in “physics on fractals.” It is a pleasure to thank Deepak Dhar and Tetsuya Hattori for useful correspondences.

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- [5] Our result does not exclude the possibility of phase transitions which do not manifest itself in decay properties of the spin-spin correlation function. An example is the chirality transition which breaks the  $Z_2 \cong O(2)/SO(2)$  symmetry.
- [6] Let  $T_c(a)$  be the critical temperature of the one-dimensional Ising ferromagnet with the Hamiltonian  $\mathcal{H} = -\sum_{i,j} |i-j|^{-a} \sigma_i \sigma_j$ . It is known that  $T_c(a)$  is finite for  $a \leq 2$  and is vanishing for  $a > 2$ . (See D. J. Thouless, Phys. Rev. **187**, 732 (1969); J. Fröhlich and T. Spencer, Commun. Math. Phys. **84**, 87 (1982); J. Cardy, J. Phys. **A14**, 1407 (1981); M. Aizenman, J. T. Chayes, L. Chayes and C. M. Newman, J. Stat. Phys. **50**, 1, 1988; J. Z. Imbrie and C. M. Newman, Commun. Math. Phys. **118**, 303 (1988), and references therein.) The discontinuity of  $T_c(a)$  at  $a = 2$  is remarkably similar to the discontinuity of  $T_c(D)$  at  $D = 2$  in the XY model studied here. Furthermore, both the models exhibit standard ferromagnetic phase transition for  $a < 2$  or  $D > 2$ , while they exhibit exotic “topological” transitions for  $a = 2$  or  $D = 2$ . However these exotic transitions may be of different nature since the Ising model with  $a = 2$  is known to have spontaneous magnetization and an intermediate phase, which are lacking in the XY model at  $D = d = 2$ .
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- [10] The standard interpretation of  $d$  in this context is the so-called spectral dimension  $d_{\text{spec}}$ . The widely accepted expectation that models with continuous symmetry has no long range order for  $d_{\text{spec}} \leq 2$  (see sect. 2.3 of [9] for example) was proved in D. Cassi, Phys. Rev. Lett. **68**, 3631 (1992) by extending the methods of [8].
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- [13] The approximation is not well controlled in the sense that one has to make these two replacements together to get a sensible result. See [7] for a critical review of the approximation.
- [14] Although the relevant fractal dimension is the one determined by the properties of the lattice Laplacian (i.e., a free random walker), it is not the spectral dimension. The dimension in (6) depends on the choice of metric  $|x - y|$  while the spectral dimension is independent of metrics.
- [15] The dimension  $D$  defined here is distinct from that appeared in (6). Nevertheless we believe that the inequality  $D < 2$  with the present definition provides the optimal criterion for the absence of phase transition. We also note that (9) may not hold in some random fractal lattices, where one cannot find the constant  $C$  which is independent of  $x$ .
- [16] It is possible to extend the theorem to the models with the Hamiltonian  $\mathcal{H} = -\sum_{u,v \in \Lambda} J_{u,v} \mathbf{s}_u \cdot \mathbf{s}_v$  with the interaction  $J_{u,v}$  satisfying  $J_{u,v} \geq 0$  and  $\sup_u \sum_v J_{u,v} \{\text{dist}(u, v)\}^2 < \infty$ . This means that moderately long ranged interactions do not change the behavior of the model for  $D < 2$ .
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